# THE PERFECT $B$-SPLINES AND A TIME-OPTIMAL CONTROL PROBLEM

BY

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ABSTRACT

Let $x_\nu = \cos(\pi\nu/n)$ $(\nu = 0, 1, \ldots, n)$. It is shown that the $B$-spline $M(x) = M(x; x_0, x_1, \ldots, x_n)$ is such that $M_n^{(n)}(x)$ has a constant absolute value $(= 2^{n-2}(n-1)!)$ in $[-1, 1]$. Its integral $f_0(x) = \int_{-1}^{x} M(t)dt$ is shown to have an optimal property that allows to solve *explicitly* a certain time-optimal control problem.

## Introduction

In [3] G. Glaeser obtains most interesting results based on the following definitions. A real-valued function $f(x)$ defined in the interval $[a, b]$ is said to be a *spline function* of degree $n$, provided that for an appropriate division $a = x_0 < x_1 < \cdots < x_k < x_{k+1} = b$ the following conditions are satisfied

1. $f(x) \in \pi_n$ in $(x_i, x_{i+1})$ $(i = 0, \cdots, k)$,
2. $f(x) \in C^{n-1}[a, b]$.

Moreover, $f(x)$ is said to be a *perfect spline*, provided that

3. $|f^{(n)}(x)| = \text{constant}$ if $a \leqq x \leqq b$, $x \neq x_1, x_2, \cdots, x_k$.

Equivalently, we can say that the $k-1$ polynomial components of $f(x)$ are of the form

$$f(x) = (-1)^i C x^n + \text{lower degree terms, in } (x_i, x_{i+1}), \quad (i = 0, \cdots, k),$$

where the constant $C$ does not depend on $i$.

THEOREM I. (G. Glaeser). *If the $2n$ reals*

$$(1) \qquad y_0^{(\nu)}, y_1^{(\nu)}, \qquad (\nu = 0, 1, \cdots, n-1)$$

*are preassigned, then the 2-point Hermite interpolation problem*

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(2) $$f^{(\nu)}(a) = y_0^{(\nu)},\ f^{(\nu)}(b) = y_1^{(\nu)}, \qquad (\nu = 0,\cdots,n-1)$$

*has a unique solution $f_0(x)$ that is a perfect spline of degree $n$ having in $(a,b)$ $k$ knots $x_1,\cdots,x_k$ such that*

(3) $$k \leqq n-1.$$

THEOREM II. (G. Glaeser). *The perfect spline interpolant $f_0(x)$ of Theorem I is the unique function that minimizes the sup-norm $\|f^{(n)}\|_\infty$ among all functions $f(x)$ such that $f^{(\nu)}(x)$ $(\nu = 0,\cdots,n-1)$ are absolutely continuous in $[a,b]$ and at the same time solve the interpolation problem* (2).

These results appeared first in Volume I of [4]. In Volume II of [4] there is a paper by R. Louboutin entitled *Sur une "bonne" partition de l'unité* [4, vol. II, pp. D1-D10], and concerned with the special case of the problem (2) when $y_0^{(\nu)} = 0$ for all $\nu$, and $y_1^{(0)} = 1$, $y_1^{(\nu)} = 0$ $(\nu = 1,\cdots,n-1)$. Since we may choose $[a,b] = [-1,1]$ without loss of generality, Louboutin's results refer to the special interpolation problem

(4) $$\begin{aligned} &f^{(\nu)}(-1) = 0, && (\nu = 0,\cdots,n-1)\\ &f(1) = 1, f^{(\nu)}(1) = 0, && (\nu = 1,\cdots,n-1). \end{aligned}$$

For this problem Louboutin obtains an *explicit expression* for the optimal solution $f_0(x)$ by establishing the following theorem.

THEOREM 3. (R. Louboutin). *Let*

(5) $$x_\nu = -\cos\frac{\pi\nu}{n} \qquad (\nu = 1,\cdots,n-1).$$

*The optimal perfect spline interpolant $f_0(x)$ of problem* (4) *has the $n-1$ knots* (5) *and*

$$|f_0^{(n)}(x)| = \text{constant} = \|f_0^{(n)}\| = 2^{n-2}(n-1)!.$$

*Moreover, $f_0(x)$ may be expressed by the formula*

$$f_0(x) = (-1)^{n-1}2^{n-2}\int_{-1}^{x}(x-t)^{n-1}\operatorname{sgn} T_n'(t)\,dt, \qquad (-1 \leqq x \leqq 1).$$

*where $T_n(x)$ is the Chebyshev polynomial.*

Since the mimeographed edition of [4] was only privately circulated, the main purpose of the present paper is to make Louboutin's results better known. We derive them below independently of Glaeser's theorem. Theorems 1 and 2 and Corollaries 1 and 2 below are Louboutin's results in somewhat different notations.

In §3 we apply them to a time-optimal control problem whose solution is expressed in terms of $f_0(x)$ (Theorem 3). In §4 we describe the behavior of $f_0(x)$ as $n \to \infty$ (Theorem 4).

## 1. The perfect $B$-splines

For convenience we let $[a,b] = [-1,1]$ and write the knots $x_i$ in decreasing order thus

$$-1 = x_n < x_{n-1} < \cdots < x_1 < x_0 = +1. \tag{1.1}$$

The corresponding divided difference of order $n$ is

$$f(x_0, x_1 \cdots, x_n) = \sum_0^n \frac{f(x_\nu)}{\omega'(x_\nu)} \tag{1.2}$$

where $\omega(x) = (x - x_0)\ (x - x_1) \cdots (x - x_n)$. It is well known that for special divisions (1.1) the divided difference (1.2) assumes simple forms. For instance, if the division (1.1) is equidistant then we know that

$$f(x_0, \cdots, x_n) = \frac{n^n}{2^n n!} \cdot \sum_0^n (-1)^\nu \binom{n}{\nu} f(x_\nu).$$

It seems less known, if at all, that an even simpler form is assumed if $x_0, x_1, \cdots, x_n$ are the abscissae where the Chebyshev polynomial $T_n(x)$ assumes its extreme values $\pm 1$. This we state as

LEMMA 1. *If*

$$x_\nu = \cos \frac{\nu\pi}{n} \quad (\nu = 0, 1, \cdots, n) \tag{1.3}$$

*then*

$$f(x_0, x_1, \cdots, x_n) = \frac{2^{n-2}}{n}(f(x_0) - 2f(x_1) + 2f(x_2) - \cdots + (-1)^{n-1}2f(x_{n-1}) + (-1)^n f(x_n)). \tag{1.4}$$

PROOF. The points (1.3) are $\pm 1$ and the $n-1$ *interior* extreme points of $T_n(x)$, hence the zeros of $T_n'(x)$. Observing that $T_n'(x) = 2^{n-1} n x^{n-1} +$ lower degree terms, we see that

$$\omega(x) = \frac{1}{2^{n-1}n}(x^2 - 1)T_n'(x). \tag{1.5}$$

The values of $\omega'(x_\nu)$ needed in (1.2), may now be easily obtained. Remembering that $T_n(x)$ satisfies the differential equation

$$(1.6)\qquad (1-x^2)T_n''(x) - xT_n'(x) + n^2T_n(x) = 0$$

we obtain from (1.5) and (1.6)

$$-2^{n-1}n\omega'(x) = \frac{d}{dx}((1-x^2)T_n'(x)) = (1-x^2)T_n''(x)$$
$$-2xT_n'(x) = -xT_n'(x) - n^2T_n(x).$$

If $x = x_\nu$ $(\nu = 1,\cdots,n-1)$ then $T_n'(x_\nu) = 0$ and therefore

$$(1.7)\qquad \omega'(x_\nu) = \frac{n}{2^{n-1}}T_n(x_\nu) = (-1)^\nu\frac{n}{2^{n-1}} \qquad (\nu = 1,\cdots,n-1).$$

If $x = 1$, then $T_n = 1$ and $T_n' = n^2$, and therefore

$$(1.8)\qquad \omega'(1) = \frac{1}{2^{n-1}n}\cdot 2n^2 = \frac{n}{2^{n-2}}, \text{ and similarly } \omega'(-1) = \frac{(-1)^n n}{2^{n-2}}.$$

Substituting the values (1.7), (1.8), into (1.2) we obtain (1.4).

It is well known that for the division (1.1) we may express the divided difference of a function $f \in C^n$ in the form

$$(1.9)\qquad f(x_0,x_1,\cdots,x_n) = \frac{1}{n!}\int_{-1}^{1} M(x)f^{(n)}(x)\,dx$$

where $M(x)$ is a certain special spline function called a $B$-spline. This fundamental relation has been known and was used for many years by a number of analysts in particular by D. V. Ionescu (see [5, Chap. III] and [6]). The linear functional $Rf = f(x_0,\cdots,x_n)$ has the property that $Rf = 0$ if $f \in \pi_{n-1}$. We may therefore apply Peano's theorem (see [2, §3.7]) and conclude that we can write

$$(1.10)\qquad \frac{1}{n!}M(x) = R_t\,\frac{(t-x)_+^{n-1}}{(n-1)!}.$$

Here we use the function $u_+ = u$ if $u \geqq 0$ and $u_+ = 0$ if $u < 0$, while the subscript "$t$" means that the operation of differencing is to be performed on the variable $t$, leaving $x$ as a parameter. By (1.2) we obtain that

$$(1.11)\qquad M(x) = \sum_0^n \frac{n(x_\nu - x)_+^{n-1}}{\omega'(x_\nu)}.$$

If we write

$$(1.12)\qquad M(x;t) = n(t-x)_+^{n-1}$$

we may also express (1.11) symbolically in Steffensen's notation in the form

$$(1.13)\qquad M(x) = M(x;x_0,x_1,\cdots,x_n).$$

Observe that (1.11) defines $M(x)$ for all real $x$ and that it has the following properties.

1° $M(x)$ is piecewise polynomial in the intervals $(-\infty, x_n), (x_n, x_{n-1}), \cdots, (x_1, x_0)$, $(x_0, \infty)$, all these polynomials being of degree $\leqq n-1$.

2° $M(x) \in C^{n-2}$

3° $M(x) = 0$ if $x < x_n$ or if $x > x_0$.

From the properties 2° and 3° it follows that

$$M^{(\nu)}(\pm 1) = 0, \qquad (\nu = 0, 1, \cdots, n-2). \tag{1.14}$$

If we substitute $f(x) = x^n$ into (1.9) we see that

$$\int_{-\infty}^{\infty} M(x)\,dx = 1. \tag{1.15}$$

A function $M(x)$ with the properties 1° and 2° is called a *spline function* of degree $n-1$ having the *knots* $x_0, x_1, \cdots, x_n$. The further property 3° together with the fact that we have precisely $n+1$ knots identifies $M(x)$ as a so-called *B-spline* (see [1, §4]). Finally (1.15) expresses the usual normalization. Concerning the descriptive properties of $B$-splines see [1, §1]. $B$-splines were the original source of the so-called Polya frequency functions as shown in [1, Part II].

In view of Lemma 1 our attention is naturally drawn to the very particular $B$-spline

$$M(x) = M(x; x_0, \cdots, x_n), \text{ where } x_\nu = \cos\frac{\nu\pi}{n} \qquad (\nu = 0, 1, \cdots, n). \tag{1.16}$$

Let us study it a little closer.

The relation (1.11) may alternatively be also written in the form

$$M(x) = (-1)^n \sum_0^n \frac{n(x - x_\nu)_+^{n-1}}{\omega'(x_\nu)}, \tag{1.17}$$

in view of the identity $(x_\nu - x)_+^{n-1} = (x_\nu - x)^{n-1} + (-1)^n (x - x_\nu)_+^{n-1}$ and the fact that

$$\sum_0^n \frac{n(x_\nu - x)^{n-1}}{\omega'(x_\nu)} = 0 \qquad \text{for all real } x,$$

the left side being a divided difference of order $n$ of an element of $\pi_{n-1}$. Differentiating (1.17) we obtain

$$M^{(n-1)}(x) = (-1)^n n! \sum_0^n \frac{(x - x_\nu)_+^0}{\omega'(x_\nu)}$$

and using (1.7) and (1.8) we get

$$M^{(n-1)}(x) = 2^{n-2}(n-1))!\{(x-x_n)_+^0 - 2(x-x_{n-1})_+^0 + \cdots + (-1)^{n-1}2(x-x_1)_+^0 + (-1)^n(x-x_0)_+^0\}. \tag{1.18}$$

Therefore

$$M^{(n-1)}(x) = \begin{cases} 2^{n-2}(n-1)! & \text{if } x_n < x < x_{n-1} \\ -2^{n-2}(n-1)! & \text{if } x_{n-1} < x < x_{n-2} \\ \vdots & \\ (-1)^{n-1}2^{n-2}(n-1)! & \text{if } x_1 < x < x_0. \end{cases} \tag{1.19}$$

We have therefore established the following

THEOREM 1. *The B-spline*

$$M(x) = M(x; x_0, x_1, \cdots, x_n) \tag{1.20}$$

*of degree $n-1$ based on the knots*

$$x_\nu = \cos\frac{\pi\nu}{n}, \qquad (\nu = 0, 1, \cdots, n) \tag{1.21}$$

*is a* perfect $B$-spline *such that*

$$|M^{(n-1)}(x)| = 2^{n-2}(n-1)! \ \text{ if } -1 < x < 1,\ x \neq x_1, \cdots, x_{n-1}. \tag{1.22}$$

From (1.14) and (1.15) we immediately obtain the following

COROLLARY 1. *If we define*

$$f_0(x) = \int_{-1}^{x} M(t)\,dt \quad in \ -1 \leqq x \leqq 1, \tag{1.23}$$

*Then $f_0(x)$ is a perfect spline function in $[-1,1]$ with knots $x_1, x_2, \cdots, x_{n-1}$, such that*

$$|f_0^{(n)}(x)| = 2^{n-2}(n-1)! \quad \text{in } (-1,1), \quad x \neq x_i. \tag{1.25}$$

*and satisfying the boundary conditions*

$$f_0^{(\nu)}(-1) = 0, \qquad (\nu = 0, 1, \cdots, n-1),$$

$$f_0(1) = 1, \quad f^{(\nu)}(1) = 0, \qquad (\nu = 1, \cdots, n-1),$$

From the properties of its derivative $M(x)$ we conclude that $f_0(x)$ is *strictly increasing* in $[-1,1]$ and that it is *convex* in $[-1,0]$ and *concave* in $[0,1]$.

## 2. An optimal property of perfect $B$-splines

So far we have established only the existence part of Glaeser's Theorem I for the special interpolation problem (4), by exhibiting its solution (1.23) explicitly (that $a = -1$ and $b = 1$ are evidently insignificant restrictions). The *unicity* of the solution will be the subject of Corollary 2 below.

THEOREM 2. *The perfect spline function* $f_0(x)$ *defined by* (1.23), *has the following optimal property: Within the class* $F_n$ *of functions* $f(x)$ *defined in* $[-1,1]$ *and satisfying the conditions*

$$f^{(\nu)}(x) \ (\nu = 0,\cdots,n-1) \quad \textit{are absolutely continuous,} \tag{2.1}$$

*and*

$$\begin{aligned} &f^{(\nu)}(-1) = 0, && (\nu = 0,1,\cdots,n-1),\\ &f(1) = 1, \quad f^{(\nu)}(1) = 0, && (\nu = 1,\cdots,n-1), \end{aligned} \tag{2.2}$$

*the function* $f_0(x)$ *is the unique function that minimizes the norm* $\|f^{(n)}\|_\infty$ *giving it its minimal value*

$$\|f_0^{(n)}\|_\infty = 2^{n-2}(n-1)! \tag{2.3}$$

REMARK. For $n = 1$ Theorem 2 reduces to the statement: If $f(x)$ is absolutely continuous in $[-1,1]$, while $f(-1) = 0$, $f(1) = 1$, then $\|f'\|_\infty \geqq \frac{1}{2}$ with equality iff $f(x) = (x+1)/2$. This is very easily shown directly. However, already for $n = 2$ Theorem 2 is far from trivial.

PROOF OF THEOREM 2. Let $f(x) \in F_n$ and assuming that

$$\|f^{(n)}\| \leqq \|f_0^{(n)}\| = 2^{n-2}(n-1)!. \tag{2.4}$$

let us show that

$$f(x) = f_0(x) \quad \text{in } [-1,1]. \tag{2.5}$$

From Theorem 1 we know that

$$f_0^{(n)}(x) = M^{(n-1)}(x) \tag{2.6}$$

is a perfect stepfunction in $[-1,1]$ which is discontinuous at the points

$$x_\nu = \cos\frac{\pi\nu}{n}, \qquad (\nu = 1,\cdots,n-1). \tag{2.7}$$

We also observe that the polynomial $T_n'(x)$ has precisely the zeros (2.7). By integrations by parts, and using (2.2), we obtain

$$\begin{aligned} \int_{-1}^{1} T'f^{(n)}dx &= \int_{-1}^{1} T'df^{(n-1)} = -\int_{-1}^{1} T''f^{(n-1)}dx = \cdots \\ &= (-1)^{n-1}\int_{-1}^{1} T_n^{(n)}f'dx \\ &= (-1)^{n-1}2^{n-1}n!\int_{-1}^{1} f'dx = (-1)^{n-1}2^{n-1}n! \end{aligned}$$

whence

$$\int_{-1}^{1}(-1)^{n-1}T_n'(x)f^{(n)}(x)\,dx = 2^{n-1}n!. \tag{2.8}$$

Since $f_0(x) \in F_n$, we may apply this to $f_0(x)$ and remembering (2.6) we obtain

$$\int_{-1}^{1}(-1)^{n-1}T_n'(x)M^{(n-1)}(x)\,dx = 2^{n-1}n! \tag{2.9}$$

Here the integrand is $> 0$ in $[-1,1]$ if $x \neq x_1,\cdots,x_{n-1}$. In fact using (1.19) we may write (2.9) as

$$\int_{-1}^{1}|T_n'(x)|2^{n-2}(n-1)!\,dx = 2^{n-1}n!. \tag{2.10}$$

From (2.4), (2.8) and (2.10) we now obtain the string of relations

$$\begin{aligned} 2^{n-1}n! &= \int_{-1}^{1}(-1)^{n-1}T_n'(x)f^{(n)}(x)\,dx \\ &\leqq \int_{-1}^{1}|T_n'(x)|\cdot 2^{n-2}(n-1)!\,dx = 2^{n-1}n!. \end{aligned} \tag{2.11}$$

The equality of the extreme terms shows that the two integrals are equal, a fact that we may write as

$$\int_{-1}^{1}\{|T_n'(x)|2^{n-2}(n-1)! - (-1)^{n-1}T_n'(x)f^{(n)}(x)\}\,dx = 0. \tag{2.12}$$

However, our assumption (2.4) shows that the integrand in (2.12) is $\geqq 0$ almost everywhere (a.e.). Therefore the vanishing of the integral (2.12) implies that the integrand must vanish a.e., whence

$$f^{(n)}(x) = (-1)^{n-1}2^{n-2}(n-1)!\ \operatorname{sgn} T_n'(x) \quad \text{a.e.} \tag{2.13}$$

However, (2.6) and (1.19) show that also

$$f_0^{(n)}(x) = (-1)^{n-1}2^{n-2}(n-1)!\ \operatorname{sgn} T_n'(x) \quad \text{a.e.}$$

Therefore $f^{(n)}(x) - f_0^{(n)}(x) = 0$ almost everywhere, so that

$$f(x) = f_0(x) + P(x) \text{ in } [-1,1], \text{ where } P(x) \in \pi_{n-1}. \tag{2.14}$$

Now the boundary properties (2.2) and (1.25) show that $P^{(\nu)}(-1) = 0$ for $\nu = 0,1,\cdots,n-1$, and therefore $P(x) = 0$ for all $x$. Finally (2.14) establishes the desired relation (2.5). This completes a proof of Theorem 2.

COROLLARY 2. *The perfect spline $f_0(x)$ of Corollary 1 is the unique perfect spline of degree $n$ in $[-1,1]$ having at most $n-1$ knots in $(-1,1)$ and such as to satisfy the boundary conditions* (1.25).

PROOF. Let $\tilde{f}(x)$ be a perfect spline of degree $n$ in $[-1,1]$ having $k$ knots, $-1 < \tilde{x}_k < \tilde{x}_{k-1} < \cdots < \tilde{x}_1 < 1$, where

$$k \leqq n-1. \tag{2.15}$$

and also satisfying the boundary conditions

$$\begin{aligned} &\tilde{f}^{(\nu)}(-1) = 0, && (\nu = 0, \cdots, n-1), \\ &\tilde{f}(1) = 1, \quad \tilde{f}^{(\nu)}(1) = 0, && (\nu = 1, \cdots, n-1). \end{aligned} \tag{2.16}$$

We observe first that

$$\tilde{M}(x) = \tilde{f}'(x) \tag{2.17}$$

is a spline function of degree $n-1$. If we extend the function $\tilde{M}(x)$ for all real $x$ by defining $\tilde{M}(x) = 0$ if $x < -1$, or if $x > 1$, then we see, by (2.16), that $\tilde{M}(x)$ is a spline function of degree $n-1$, of class $C^{n-2}$ for all real $x$, having the knots

$$-1, \tilde{x}_k, \cdots, \tilde{x}_1, 1 \tag{2.18}$$

and vanishing in the complement of $(-1,1)$. It is known that such spline functions must have *at least* $n+1$ knots (See [1, Theorem 4]). By (2.18) we must therefore have $k+2 \geqq n+1$ or $k \geqq n-1$. Now our assumption (2.15) shows that $k = n-1$. Again Theorem 4 of [1] shows that $\tilde{M}(x)$ is a $B$-spline that (in our previous notation) we may write as

$$\tilde{M}(x) = M(x; 1, \tilde{x}_1, \cdots, \tilde{x}_{n-1}, -1). \tag{2.19}$$

Moreover, let

$$|\tilde{M}^{(n-1)}(x)| = |\tilde{f}^{(n)}(x)| = C \qquad \text{for all } x \neq \tilde{x}_\nu. \tag{2.20}$$

We can now establish for $\tilde{f}(x)$ an optimal property by repeating the proof of Theorem 2. This property will show that

$$C = \inf_{f \in F_n} \|f^{(n)}\|,$$

while $\tilde{f}(x)$ is the unique extremizing function. In this proof, which we omit, the role of $T_n'(x)$ would be played by the polynomial

$$\tilde{T}'(x) = (x - \tilde{x}_1) \cdots (x - \tilde{x}_{n-1}).$$

This optimal property of $\tilde{f}(x)$ having been established, we conclude by Theorem 2 that $C = 2^{n-2}(n-1)!$ and that $\tilde{f}(x) = f_0(x)$ in $[-1,1]$.

## 3. A time-optimal control problem

The problem we have in mind is as follows.

PROBLEM 1. *A particle $F$ moves on the y-axis such that $y = F(t)$ and that the velocities of different orders*

$$F^{(\nu)}(t), \qquad (\nu = 0,1,\cdots,n-1)$$

*are all absolutely continuous, while the* n*th velocity $F^{(n)}(t)$ satisfies at all times the inequality*

$$|F^{(n)}(t)| \leqq A \qquad (A \text{ is preassigned}). \tag{3.1}$$

*We assume that the particle $F$ starts from rest at $y = 0$ at the time $t = 0$, i.e.*

$$F^{(\nu)}(0) = 0 \qquad (\nu = 0,1,\cdots,n-1), \tag{3.2}$$

*and that $F$ reaches the point $y = l$ also at rest at the time $t = T$ $(>0)$, i.e.*

$$F(T) = l,\ F^{(\nu)}(T) = 0 \quad (\nu = 1,\cdots,n-1). \tag{3.3}$$

*We are to find the shortest time $T_0$ during which this motion can be performed and are to describe the nature of this optimal motion.*

SOLUTION. If we recall that the perfect spline $f_0(x)$, defined by (1.23), satisfies the boundary conditions (1.25), it seems clear that

$$F(t) = lf_0\left(\frac{2t}{T} - 1\right), \qquad 0 \leqq t \leqq T, \tag{3.4}$$

is a function satisfying the conditions (3.2) and (3.3). It is likewise clear that $F(t)$ is a perfect spline in $[0,T]$. Also, by (2.3), that

$$\| F^{(n)} \| = l\left(\frac{2}{T}\right)^n 2^{n-2}(n-1)!. \tag{3.5}$$

If we also wish to meet the condition (3.1), as we must, we obtain the inequality

$$l\left(\frac{2}{T}\right)^n 2^{n-2}(n-1)! \leqq A.$$

Selecting here the least such $T$, producing the *equality* here, we obtain

$$T_0 = 2(2^{n-2}(n-1)!\, l/A)^{1/n}. \tag{3.6}$$

THEOREM 3. *The motion corresponding to*

(3.7) $$F_0(t) = lf_0\left(\frac{2t}{T_0} - 1\right), \qquad 0 \leqq t \leqq T_0,$$

*is the optimal one and* $T_0$, *given by* (3.6), *is the least time in which the motion of Problem* 1 *can be performed.*

We prepare a proof of Theorem 3 by first generalizing Theorem 2 in an obvious way that may be described as the introduction of superfluous parameters.

COROLLARY 3. *Among all functions* $G(t)$ *defined in* $[0, T_0]$ *and satisfying the conditions of absolute continuity of the derivatives and the interpolatory conditions*

(3.8) $$\begin{aligned} &G^{(\nu)}(0) = 0, && (\nu = 0, \cdots, n-1), \\ &G(T_0) = l, \quad G^{(\nu)}(T_0) = 0, && (\nu = 1, \cdots, n-1), \end{aligned}$$

*the function* $F_0(t)$, *defined by* (3.7), *is the unique function minimizing the norm* $\| G^{(n)} \|$ *and giving it its minimal value*

(3.9) $$\min \| G^{(n)} \| = \| F_0^{(n)} \| = l\left(\frac{2}{T_0}\right)^n 2^{n-2}(n-1)! = A.$$

PROOF OF COROLLARY 3. Let $G(t)$ satisfy (3.8). Then evidently the function

$$f(x) = l^{-1}G\left(\frac{T_0}{2}(x+1)\right), \qquad (-1 \leqq x \leqq 1),$$

satisfies the assumptions (2.1) and (2.2) of Theorem 2 while

$$\| f^{(n)} \| = l^{-1}\left(\frac{T_0}{2}\right)^n \| G^{(n)} \|.$$

By Theorem 2 we conclude that this is $\geqq 2^{n-2}(n-1)!$ and therefore

$$\| G^{(n)} \| \geqq l\left(\frac{2}{T_0}\right)^n 2^{n-2}(n-1)! = A,$$

while $\| F_0^{(n)} \| = A$. The unicity of $F_0(t)$ also follows from Theorem 2.

PROOF OF THEOREM 3. Let $F(t)$ be a motion in the time interval $[0, T]$ satisfying the conditions (3.1), (3.2) and (3.3) of Problem 1 and let us show that

(3.10) $$T \geqq T_0,$$

with equality iff $F(t)$ is the motion (3.7). To show this we consider the motion

$$G(t) = F(tT/T_0), \qquad 0 \leqq t \leqq T_0,$$

which evidently satisfies the conditions (3.8) of Corollary 3. However

$$G^{(n)}(t) = \left(\frac{T}{T_0}\right)^n F^{(n)}(tT/T_0)$$

and therefore by Corollary 3 and (3.1)

$$A \leqq \| G^{(n)} \| = \left(\frac{T}{T_0}\right)^n \| F^{(n)} \| \leqq \left(\frac{T}{T_0}\right)^n A. \tag{3.11}$$

Hence $A \leqq (T/T_0)^n A$ and (3.10) is established. Moreover, again by Corollary 3, if $T = T_0$ then (3.11) shows that $\| G^{(n)} \| = A$ and therefore $F(t) = F_0(t)$.

REMARK. The solution of the optimal control Problem 1 for the special case when $n = 2$ is well known. For its discussion from the point of view of general optimal control theory we refer to [7, 23–27] and [8, 233–236].

## 4. The behavior of the perfect spline $f_0(x)$ as $n \to \infty$

The function $f_0(x)$ defined by (1.23) is a distribution function whose frequency function is defined by (1.20) and (1.21). In order to indicate their dependence on the integer $n$ we write

$$f_0(x) = f_{0,n}(x) \text{ and } M(x) = M_n(x). \tag{4.1}$$

Their behavior as $n \to \infty$ is described as follows.

THEOREM 4. *The distributions described by the functions* (4.1) *converge, as $n \to \infty$, to the unit point-mass placed at the origin. This means that for arbitrarily small positive $\delta$ we have*

$$\lim_{n\to\infty} M_n(x) = 0 \textit{ uniformly in } x \textit{ if } |x| \geqq \delta, \tag{4.2}$$

*and therefore*

$$\lim_{n\to\infty} f_{0,n}(x) = 0 \textit{ uniformly in } -1 \leqq x \leqq -\delta . \tag{4.3}$$

$$\lim_{n\to\infty} (1 - f_{0,n}(x)) = 0 \textit{ uniformly in } \delta \leqq x \leqq 1. \tag{4.4}$$

It was pointed out in [1, Footnote on page 74] that the mean and variance of the general $B$-spline

$$M(x) = M(x; x_0, x_1, \cdots, x_n) \tag{4.5}$$

are given by

$$\mu_1 = \int_{-\infty}^{\infty} x M(x) dx = \frac{1}{n+1} \sum_{0}^{n} x_\nu \tag{4.6}$$

and

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu_1)^2 M(x)\,dx = \frac{1}{(n+1)^2(n+2)} \sum_{i<j} (x_i - x_j)^2. \tag{4.7}$$

From these formulas we derive the following general

LEMMA 2. *Let*

$$\tilde{M}_n(x) = M(x; x_{0,n}, x_{1,n}, \cdots, x_{n,n}) \qquad (n = 1,2,\cdots) \tag{4.8}$$

*be a sequence of B-splines subject only to the restriction that its knots are all in* $[-1,1]$, *i.e.*

$$-1 \leqq x_{n,n} \leqq x_{n-1,n} \leqq \cdots \leqq x_{0,n} \leqq 1, \qquad x_{n,n} < x_{0,n}. \tag{4.9}$$

*Let* $\tilde{\sigma}_n$ *denote the standard deviation of* $\tilde{M}_n(x)$. *Then*

$$\tilde{\sigma}_n < \frac{1}{\sqrt{n}} \text{ for all } n. \tag{4.10}$$

PROOF OF LEMMA 2. It is clear that the right side of (4.7), if viewed as a quadratic function of the variables $x_\nu$, all contained in $[-1,1]$, will be maximized if, as nearly as possible, one half of the points $x_\nu$ are at $-1$ and the other half at $+1$. The details depend on the parity of $n+1$.

1. $n+1 = 2k$ *is even*. We can then place $k$ points at each of the two endpoints to obtain

$$\max \sum_{i<j} (x_k - x_j)^2 = 2^2 k \cdot k.$$

By (4.7) we have

$$\max \sigma^2 = \frac{1}{2k+1} = \frac{1}{n+2} < \frac{1}{n}.$$

2. $n+1 = 2k+1$ *is odd*. We must place $k$ points at one endpoint and $k+1$ points at the other to obtain

$$\max \sum_{i<j} (x_i - x_j)^2 = 2^2 k(k+1),$$

and (4.7) shows that

$$\max \sigma^2 = \frac{2k}{(2k+1)^2} = \frac{n}{(n+1)^2} < \frac{1}{n}.$$

The inequality (4.10) evidently follows.

PROOF OF THEOREM 4. We return to our sequence of perfect *B*-splines $M_n(x)$ having variances $\sigma_n^2 < 1/n$ by Lemma 2. By (1.21) and (4.6) their means are $= 0$. Moreover, $M_n(x)$ is an even function that is decreasing in the interval $[0,1]$. Let

$$\delta = 2\varepsilon.$$

By comparison of areas and Chebyshev's inequality we see that

$$\varepsilon M_n(2\varepsilon) < \int_\varepsilon^1 M_n(x)\,dx = \frac{1}{2}\int_{|x|\geqq\varepsilon} M_n(x)\,dx \leqq \frac{1}{2}\frac{\sigma_n^2}{\varepsilon^2} < \frac{2}{\delta^2 n}$$

whence

$$M_n(\delta) < \frac{4}{\delta^3 n} \to 0 \text{ as } n \to \infty,$$

and Theorem 4 is thereby established.

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